Tilting Cohen-Macaulay representations

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Aim : Study Cohen-Macaulay modules by using tilting theory and cluster tilting theory

Throughout k is an algebraically closed field of characteristic zero for simplicity

Part 1 : Tilting theory (Derived category)
$\Longrightarrow$ graded Cohen-Macaulay modules

Theorem [Gabriel '72]

Q : an acyclic connected quiver, kQ : the path algebra of Q

• kQ is representation-finite (i.e. there are only finitely many isoclasses of indecomposable kQ-modules)

$\Longleftrightarrow$ Q is a Dynkin quiver

$$A_n\ (n \geq 1) \qquad \bullet - \bullet - \bullet - \bullet - \cdots - \bullet - \bullet - \bullet$$

$$D_n\ (n \geq 4) \qquad \begin{array}{ccccccccccc} & & \bullet & & & & & & & & \\ & & | & & & & & & & & \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \cdots & - & \bullet & - & \bullet \end{array}$$

$$E_n\ (n = 6,7,8) \qquad \begin{array}{ccccccccccc} & & & & \bullet & & & & & & \\ & & & & | & & & & & & \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \cdots & - & \bullet & - & \bullet \end{array}$$

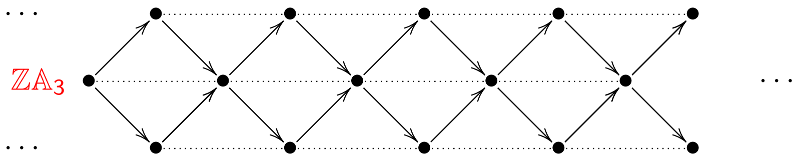
• In this case, $\exists$ a bijection between isoclasses of indecomposable kQ-modules and positive roots

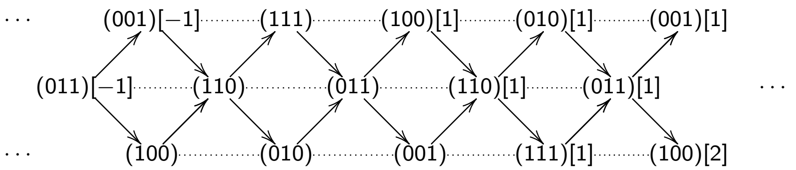
- mod kQ : the category of finitely generated kQ-modules
- $\mathrm{D^b}(\mathrm{mod}\, kQ)$: the bounded derived category of mod kQ

Theorem [Happel '87]

Q : a Dynkin quiver
The Auslander-Reiten quiver of $\mathrm{D^b}(\mathrm{mod}\, kQ)$ is $\mathbb{Z}Q$

- $Q = \mathbb{A}_3 = [\bullet \longrightarrow \bullet \longrightarrow \bullet]$





Cohen-Macaulay representations

- R : a Gorenstein ring $\Longleftrightarrow$ a Noetherian ring satisfying inj.dim $R_R < \infty$ and inj.dim ${}_R R < \infty$
- $X \in \mathrm{mod}\, R$ is Cohen-Macaulay $\Longleftrightarrow$ $\mathrm{Ext}^i_R(X, R) = 0$ for all $i > 0$
- CM R : the category of Cohen-Macaulay R-modules

Example

R : a commutative local Gorenstein ring
- $X \in \mathrm{mod}\, R$ is Cohen-Macaulay $\Longleftrightarrow$ depth $X = \dim R$ or $X = 0$

Definition [Auslander-Bridger]

$\underline{\mathrm{CM}}\, R$: the stable category
- The objects are the same as CM R
- $\mathrm{Hom}_{\underline{\mathrm{CM}}\, R}(X, Y) := \mathrm{Hom}_R(X, Y)/P(X, Y)$, where $P(X, Y)$ consists of morphisms factoring through projective R-modules

Properties [Happel, Buchweitz, Eisenbud]

- $\underline{\mathrm{CM}}\, R$ has a canonical structure of a triangulated category
- $\exists$ a triangle equivalence $\underline{\mathrm{CM}}\, R \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)/\mathrm{K}^{\mathrm{b}}(\operatorname{proj} R)$

$\operatorname{proj} R$: the category of finitely generated projective R-modules
$\mathrm{K}^{\mathrm{b}}(\operatorname{proj} R)$: the bounded homotopy category of $\operatorname{proj} R$

- R : a hypersurface

$\Longrightarrow$ $\underline{\mathrm{CM}}\, R$ is described by matrix factorizations

More generally

- G : an abelian group
- R : a G-graded Gorenstein ring
- $\mathrm{CM}^G R$: the category of G-graded Cohen-Macaulay R-modules
- $\underline{\mathrm{CM}}^G R$: the stable category is defined similarly

- $\underline{\mathrm{CM}}^G R$ has a canonical structure of a triangulated category

- Other properties also hold after suitable modifications

Example

- $R := k[x]/(x^n)$ with $\deg x = 1$
- R is selfinjective ($\Longleftrightarrow$ $\operatorname{inj.dim} R = 0$) and hence Gorenstein
- $\mathrm{CM}^{\mathbb{Z}} R = \operatorname{mod}^{\mathbb{Z}} R$
- Indecomposable objects in $\operatorname{mod} R$ are $X_i = k[x]/(x^i)$ $(1 \le i \le n)$
- Indecomposable objects in $\operatorname{mod}^{\mathbb{Z}} R$ are $X_i(j)$ $(1 \le i \le n,\ j \in \mathbb{Z})$
- For $n = 4$, the Auslander-Reiten quiver of $\underline{\operatorname{mod}}^{\mathbb{Z}} R$ is

$$\begin{tikzcd}
\cdots & & X_1(-2) \arrow[dr] \arrow[rr, dotted, no head] & & X_1(-1) \arrow[dr] \arrow[rr, dotted, no head] & & X_1 \arrow[dr] \arrow[rr, dotted, no head] & & X_1(1) \arrow[dr] \arrow[rr, dotted, no head] & & X_1(2) & \\
& X_2(-2) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & X_2(-1) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & X_2 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & X_2(1) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & X_2(2) \arrow[ur] \arrow[dr] & & \cdots \\
\cdots & & X_3(-1) \arrow[ur] \arrow[rr, dotted, no head] & & X_3 \arrow[ur] \arrow[rr, dotted, no head] & & X_3(1) \arrow[ur] \arrow[rr, dotted, no head] & & X_3(2) \arrow[ur] \arrow[rr, dotted, no head] & & X_3(3) &
\end{tikzcd}$$

This is $\mathbb{Z}\mathbb{A}_3$, the Auslander-Reiten quiver of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} k\mathbb{A}_3)$

$\exists$ a triangle equivalence $\underline{\operatorname{mod}}^{\mathbb{Z}}(k[x]/(x^n)) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod} k\mathbb{A}_{n-1})$

Theorem [Buchweitz-Greuel-Schreyer '87]

R : a commutative complete local Gorenstein k-algebra

• R is CM-finite (i.e. there are only finitely many isoclasses of indecomposable objects in $\mathrm{CM}\,R$) $\iff$ R is a simple singularity

$$R = k[[x, y, z_2 \dots, z_d]]/(f),\ f = \begin{cases} x^{n+1} + y^2 + z_2^2 + \cdots + z_d^2 & A_n \\ x^{n-1} + xy^2 + z_2^2 + \cdots + z_d^2 & D_n \\ x^4 + y^3 + z_2^2 + \cdots + z_d^2 & E_6 \\ x^3y + y^3 + z_2^2 + \cdots + z_d^2 & E_7 \\ x^5 + y^3 + z_2^2 + \cdots + z_d^2 & E_8. \end{cases}$$

Theorem [Geigle-Lenzing '91, Kajiura-Saito-Takahashi '07]

$R = k[x, y, z_2]/(f)$: a simple singularity with $\dim R = 2$ and $\mathbb{Z}$-grading given by

R and Q	$A_n : x^{n+1} - yz_2$	D_n	E_6	E_7	E_8
$\deg x$	1	2	3	4	6
$\deg y$	p	$n-2$	4	6	10
$\deg z_2$	$n+1-p$	$n-1$	6	9	15

$\Longrightarrow \exists$ a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}\, R \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, kQ)$

Gabriel's Theorem $\longleftrightarrow$ Buchweitz-Greuel-Schreyer's Theorem

Tilting theory

$\mathcal{T}$: a triangulated category

Definition [Rickard '89]

$U \in \mathcal{T}$ is a tilting object $\iff$

• $\forall i \neq 0$, $\mathrm{Hom}_{\mathcal{T}}(U, U[i]) = 0$

• U generates $\mathcal{T}$ as a thick subcategory

• $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ has a tilting object $\Lambda = (\cdots \to 0 \to 0 \to \Lambda \to 0 \to \cdots)$

Proposition [Keller '94]

$\mathcal{T}$: an algebraic triangulated category (e.g. $\underline{\mathrm{CM}}^{G}\, R$)

$U \in \mathcal{T}$: a tilting object

$\Longrightarrow \exists$ a triangle equivalence $\mathcal{T} \simeq \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ for $\Lambda := \mathrm{End}_{\mathcal{T}}(U)$

• If moreover $\mathrm{gl.dim}\,\Lambda < \infty$, then $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda) = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ and we have a triangle equivalence $\mathcal{T} \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$

Dimension zero

Theorem [Yamaura '13]

- R : a $\mathbb{Z}$-graded finite dimensional selfinjective k-algebra
- Assume $R_i = 0$ for $i < 0$ and $\mathrm{gl.dim}\, R_0 < \infty$

$\Longrightarrow$ $\underline{\mathrm{mod}}^{\mathbb{Z}} R$ has a tilting object $\bigoplus_{i\geq 0} R(i)_{\geq 0}$
where $X_{\geq 0} := \bigoplus_{i\geq 0} X_i$ for $X \in \mathrm{mod}^{\mathbb{Z}} R$

- $R = k[x]/(x^n)$ with $\deg x = 1$

$\Longrightarrow$ $\exists$ a triangle equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}}(k[x]/(x^n)) \simeq \mathrm{D^b}(\mathrm{mod}\, k\mathbb{A}_{n-1})$

Corollary [Happel '87]

- Λ : a finite dimensional k-algebra with $\mathrm{gl.dim}\,\Lambda < \infty$
- $T(\Lambda) = \Lambda \oplus D\Lambda$: the trivial extension algebra (D : the k-dual)

$(x_1, f_1)\cdot(x_2, f_2) = (x_1x_2, x_1f_2 + f_1x_2)$ for $x_1, x_2 \in \Lambda$, $f_1, f_2 \in D\Lambda$
$T(\Lambda)_0 = \Lambda$, $T(\Lambda)_1 = D\Lambda$, $T(\Lambda)_i = 0$ for $i \neq 0, 1$
$\Longrightarrow$ $\exists$ a triangle equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}}\, T(\Lambda) \simeq \mathrm{D^b}(\mathrm{mod}\,\Lambda)$

Dimension one

Theorem [Buchweitz-I-Yamaura (arXiv:1803.05269)]

- R : a commutative Gorenstein ring with $\dim R = 1$
- Assume $R_i = 0$ for $i < 0$, $R_0 = k$ and R is reduced

$\Longrightarrow$ $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ has a tilting object $\bigoplus_{i=1}^{n} R(i)_{\geq 0}$ ($n \gg 0$)

- There is a result for non-reduced case

Example (cf. [Dieterich-Wiedemann, Araya])

$R = k[x, y]/(f)$: a simple singularity with $\dim R = 1$ and $\mathbb{Z}$-grading given by

R	A_{2n-1}	A_{2n}	D_{2n}	D_{2n+1}	E_6	E_7	E_8
$\deg x$	1	2	1	2	3	2	3
$\deg y$	n	$2n+1$	$n-1$	$2n-3$	4	3	5
Q	D_{n+1}	A_{2n}	D_{2n}	A_{4n-1}	E_6	E_7	E_8

$\Longrightarrow$ $\exists$ a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq \mathrm{D^b}(\mathrm{mod}\, kQ)$

Geigle-Lenzing complete intersection (arXiv:1409.0668)

[joint work with Herschend, Minamoto, Oppermann]

- $C := k[T_0, \dots, T_d]$: a polynomial algebra over a field k ($d \geq 0$)
- $\ell_1, \dots, \ell_n \in C$: linear forms ($n \geq 0$)
- $p_1, \dots, p_n \geq 2$: integers

Definition

- $R := C[X_1, \dots, X_n]/(X_i^{p_i} - \ell_i \mid 1 \leq i \leq n)$: a ring
- $\mathbb{L} := \langle \vec{c}, \vec{x}_1, \dots, \vec{x}_n \rangle / \langle p_i \vec{x}_i - \vec{c} \mid 1 \leq i \leq n \rangle$: an abelian group

Properties

- R is $\mathbb{L}$-graded by $\deg T_j := \vec{c}$ and $\deg X_i := \vec{x}_i$
- $\mathbb{L}$ is an abelian group of rank one
- R is a complete intersection ring with $\dim R = d + 1$
- R has the a-invariant $\vec{\omega} := (n - d - 1)\vec{c} - \sum_{i=1}^{n} \vec{x}_i \in \mathbb{L}$

i.e. $\mathrm{Ext}_R^{d+1}(k, R(\vec{\omega})) \simeq k$ in $\mathrm{mod}^{\mathbb{L}} R$

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Definition

$(R, \mathbb{L})$: a Geigle-Lenzing complete intersection $\iff$
$\forall$ at most $d+1$ elements from $\ell_1, \dots, \ell_n$ are linearly independent

- $n \leq d + 1 \iff R$ is a polynomial algebra
- $n = d + 2 \iff R$ is a hypersurface

$R \simeq k[X_1, \dots, X_n]/(\alpha_1 X_1^{p_1} + \cdots + \alpha_n X_n^{p_n})$

Classical case $d = 1$ [Geigle-Lenzing '87, '91]

- $(R, \mathbb{L})$ was used to study the weighted projective line

$R \simeq k[X_1, \cdots, X_n]/(X_i^{p_i} - \alpha_{i1} X_1^{p_1} - \alpha_{i2} X_2^{p_2})_{3 \leq i \leq n}$

- $(R, \mathbb{L})$ is domestic $\iff n - 2 \leq \frac{1}{p_1} + \cdots + \frac{1}{p_n}$

In this case, the Veronese subring $R^{(\vec{\omega})}$ is a simple singularity with $\dim R^{(\vec{\omega})} = 2$ and $\mathrm{CM}^{\mathbb{L}} R \simeq \mathrm{CM}^{\mathbb{Z}} R^{(\vec{\omega})}$

$(p_1, \dots, p_n)$	$n \leq 2$	$(2,2,p)$	$(2,3,3)$	$(2,3,4)$	$(2,3,5)$
$R^{(\vec{\omega})}$	$A_{p_1 + \cdots + p_n - 1}$	D_{p+2}	E_6	E_7	E_8

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Theorem [HIMO '14, Futaki-Ueda '11 for $n = d + 2$, Kussin-Lenzing-Meltzer '13 for $d = 1,\ n = 3$]

$(R, \mathbb{L})$: a Geigle-Lenzing complete intersection
$\Longrightarrow$ $\exists$ a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{L}} R \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{CM}})$

- $\mathbb{L}$ is a poset : $\vec{x} \geq \vec{y} \iff \vec{x} - \vec{y} \in \langle \vec{c}, \vec{x}_1, \ldots, \vec{x}_n \rangle_{\text{monoid}}$
- $A^{\mathrm{CM}} := (R_{\vec{x}-\vec{y}})_{0 \leq \vec{x}, \vec{y} \leq d\vec{c}+2\vec{\omega}}$

This is a k-algebra by product in R and matrix multiplication

- $n = d + 2 \Longrightarrow A^{\mathrm{CM}} = \bigotimes_{i=1}^{n} k\mathbb{A}_{p_i - 1}$
- $n = d + 3$ and $p_1 = \cdots = p_n = 2$

$\Longrightarrow$ A^{CM} is given by the following quiver with 2 relations

$$\begin{tikzcd}[row sep=small]
 & \vec{x}_1 \arrow[ddr] & \\
 & \vec{x}_2 \arrow[dr] & \\
0 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \vdots & \vec{c} \\
 & \vec{x}_{n-1} \arrow[ur] & \\
 & \vec{x}_n \arrow[uur] &
\end{tikzcd}$$

Part 2 : Cluster category (Cluster tilting theory)
$\Longrightarrow$ ungraded Cohen-Macaulay modules

- Cluster categories were introduced in categorification of Fomin-Zelevinsky cluster algebras
- [Buan-Marsh-Reineke-Reiten-Todorov '06] for $\Lambda = kQ$
- [Amiot '09, Guo '11 based on Keller '05] for general Λ

Definition

$\mathcal{T}$: an additive category
$F : \mathcal{T} \simeq \mathcal{T}$: an automorphism

- The orbit category $\mathcal{T}/F$ has the same objects as $\mathcal{T}$
- $\mathrm{Hom}_{\mathcal{T}/F}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, F^i(Y))$

where the composition is defined naturally

Recall

- $\exists$ a triangle equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}}(k[x]/(x^n)) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k\mathbb{A}_{n-1})$

$$
\begin{tikzcd}[column sep=small]
\cdots & & X_1(-2) \arrow[dr] \arrow[rr, dotted, dash] & & X_1(-1) \arrow[dr] \arrow[rr, dotted, dash] & & X_1 \arrow[dr] \arrow[rr, dotted, dash] & & X_1(1) \arrow[dr] \arrow[rr, dotted, dash] & & X_1(2) & \\
& X_2(-2) \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & X_2(-1) \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & X_2 \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & X_2(1) \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & X_2(2) \arrow[ur] \arrow[dr] & & \cdots \\
\cdots & & X_3(-1) \arrow[ur] \arrow[rr, dotted, dash] & & X_3 \arrow[ur] \arrow[rr, dotted, dash] & & X_3(1) \arrow[ur] \arrow[rr, dotted, dash] & & X_3(2) \arrow[ur] \arrow[rr, dotted, dash] & & X_3(3) &
\end{tikzcd}
$$

$$\xrightarrow{(1)}$$

$$
\begin{tikzcd}[column sep=small]
\cdots & & (001)[-1] \arrow[dr] \arrow[rr, dotted, dash] & & (111) \arrow[dr] \arrow[rr, dotted, dash] & & (100)[1] \arrow[dr] \arrow[rr, dotted, dash] & & (010)[1] \arrow[dr] \arrow[rr, dotted, dash] & & (001)[1] & \\
& (011)[-1] \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & (110) \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & (011) \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & (110)[1] \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & (011)[1] \arrow[ur] \arrow[dr] & & \cdots \\
\cdots & & (100) \arrow[ur] \arrow[rr, dotted, dash] & & (010) \arrow[ur] \arrow[rr, dotted, dash] & & (001) \arrow[ur] \arrow[rr, dotted, dash] & & (111)[1] \arrow[ur] \arrow[rr, dotted, dash] & & (100)[2] &
\end{tikzcd}
$$

$$\xleftarrow{\tau}$$

- $(1) \simeq \tau^{-1}$ holds
- Their orbit categories are also equivalent

$$
\begin{array}{ccc}
\underline{\mathrm{mod}}^{\mathbb{Z}}(k[x]/(x^n))/(1) & \xrightarrow{\simeq} & \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k\mathbb{A}_{n-1})/\tau \\
\| & & \| \\
\underline{\mathrm{mod}}\,(k[x]/(x^n)) & & \mathrm{C}_1(k\mathbb{A}_{n-1})
\end{array}
$$

Serre functor

- $\mathcal{T}$: a k-linear Hom-finite triangulated category

Definition

- An automorphism $\mathbb{S} : \mathcal{T} \to \mathcal{T}$ is a Serre functor $\Longleftrightarrow$

$\exists$ a bifunctorial isomophism $\mathrm{Hom}_{\mathcal{T}}(X, Y) \simeq D\mathrm{Hom}_{\mathcal{T}}(Y, \mathbb{S}X)$

- [Happel] Serre functor $\Longrightarrow$ Auslander-Reiten theory in $\mathcal{T}$

Example

- X : a smooth projective variety over k with $\dim X = d$

Then $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ has a Serre functor $- \otimes \omega_X[d]$

- [Happel] Λ : a finite dimensional k-algebra with $\mathrm{gl.dim}\, \Lambda < \infty$

Then $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \Lambda)$ has a Serre functor $\nu := - \overset{\mathbf{L}}{\otimes}_{\Lambda} D\Lambda$

- [Auslander] R : a Gorenstein isolated singularity, $\dim R = d + 1$

Then $\underline{\mathrm{CM}}\, R$ is d-Calabi-Yau (i.e. $[d]$ is a Serre functor)

Cluster category

- $d \geq 1$
- Λ : a finite dimensional k-algebra with $\mathrm{gl.dim}\,\Lambda \leq d$
- $\nu_d := \nu \circ [-d] : \mathrm{D^b}(\mathrm{mod}\,\Lambda) \simeq \mathrm{D^b}(\mathrm{mod}\,\Lambda)$ (e.g. $\tau = \nu_1$)
- $\mathrm{C}_d^\circ(\Lambda) := \mathrm{D^b}(\mathrm{mod}\,\Lambda)/\nu_d$

This does not have a structure of a triangulated category in general

Theorem [AGK]

- $\exists$ a triangulated category $\mathrm{C}_d(\Lambda)$ (the d-cluster category) containing $\mathrm{C}_d^\circ(\Lambda)$ as a full subcategory such that the composition $\mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{C}_d^\circ(\Lambda) \subset \mathrm{C}_d(\Lambda)$ is a triangle functor
- If $\mathrm{C}_d(\Lambda)$ is Hom-finite, then it is d-Calabi-Yau

Cluster tilting

- Appeared in higher dimensional Auslander-Reiten theory

Definition [I '07]

$d \geq 1$

$\mathcal{T}$: a triangulated category or an exact category

$M \in \mathcal{T}$ is a d-cluster tilting object

$\Longleftrightarrow$ The following conditions for $X \in \mathcal{T}$ are equivalent

- X is a direct summands of $M^{\oplus n}$ for some $n \geq 0$
- $1 \leq \forall i \leq d-1$, $\mathrm{Hom}_{\mathcal{T}}(M, X[i]) = 0$
- $1 \leq \forall i \leq d-1$, $\mathrm{Hom}_{\mathcal{T}}(X, M[i]) = 0$

Example

- [AG] If $\mathrm{C}_d(\Lambda)$ is Hom-finite, then $\Lambda \in \mathrm{C}_d(\Lambda)$ is d-cluster tilting
- R is CM-finite $\Longleftrightarrow$ CM R has a 1-cluster tilting object
- [I] R : a Gorenstein isolated singularity with $\dim R = d+1$

$M \in \mathrm{CM}\,R$ is a d-cluster tilting object

$\Longleftrightarrow$ $\mathrm{End}_R(M)$ is a non-commutative crepant resolution of R

Preprojective algebra

- Λ : a finite dimensional k-algebra with $\mathrm{gl.dim}\,\Lambda \le d$
- $E := \mathrm{Ext}^d_\Lambda(D\Lambda, \Lambda)$ has a Λ-bimodule structure

Definition [I-Oppermann, Keller]

- $\Pi = \Pi_{d+1}(\Lambda) := T_\Lambda(E)$: the $(d+1)$-preprojective algebra
- This is $\mathbb{Z}$-graded by $\Pi_i = E^{\otimes_\Lambda i}$ for $i \ge 0$

- $\Pi_0 = \Lambda$ and $\Pi_i = H^0(\nu_d^{-i}(\Lambda))$ for $i \ge 0$
- $\Pi = \mathrm{End}_{\mathrm{C}_d(\Lambda)}(\Lambda)$

Classical case $d = 1$

- Λ : a finite dimensional hereditary k-algebra (i.e. $\mathrm{gl.dim}\,\Lambda \le 1$)

$\iff \Lambda = kQ$ for an acyclic quiver Q
- $\overline{Q}$: the double of Q

$$\overline{\mathbb{A}}_n : \bullet \underset{a_1^*}{\overset{a_1}{\rightleftarrows}} \bullet \underset{a_2^*}{\overset{a_2}{\rightleftarrows}} \bullet \underset{a_3^*}{\overset{a_3}{\rightleftarrows}} \cdots \underset{a_{n-2}^*}{\overset{a_{n-2}}{\rightleftarrows}} \bullet \underset{a_{n-1}^*}{\overset{a_{n-1}}{\rightleftarrows}} \bullet$$

- $\Pi_2(kQ)$ is isomorphic to $k\overline{Q}/\langle \sum_{a \in Q_1}(aa^* - a^*a)\rangle$

- Λ : a finite dimensional k-algebra with $\mathrm{gl.dim}\,\Lambda \le d$
- $\Pi_{d+1}(\Lambda)$ enjoys nice homological properties under the condition :

Definition

Λ : d-hereditary $\iff \forall i \in \mathbb{Z}$, $\nu_d^i(\Lambda)$ is concentrated in degrees $d\mathbb{Z}$

Dichotomy Theorem [Herschend-I-Oppermann '14]

Assume that Λ is ring-indecomposable

Λ is d-hereditary $\iff$ One of the following holds
- $\exists$ a d-cluster tilting object $M \in \mathrm{mod}\,\Lambda$ (d-representation-finite)
- $\forall i \ge 0$, $\nu_d^{-i}(\Lambda) \in \mathrm{mod}\,\Lambda$ (d-representation-infinite)

Classical case $d = 1$

A finite dimensional k-algebra Λ is
- 1-representation-finite $\iff \Lambda = kQ$ for a Dynkin quiver Q
- 1-representation-infinite $\iff \Lambda = kQ$ for a non-Dynkin quiver Q

d-representation-finite case

- Λ : a d-representation-finite k-algebra
- $\Pi = \Pi_{d+1}(\Lambda)$

Theorem [IO '13, Amiot '09 for $d = 1$]

- Π is a finite dimensional selfinjective k-algebra
- $\exists$ a triangle equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}}\,\Pi \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{End}_{\underline{\mathrm{mod}}\,\Lambda}(\Pi))$
- $\exists$ a triangle equivalence $\underline{\mathrm{mod}}\,\Pi \simeq \mathrm{C}_{d+1}(\mathrm{End}_{\underline{\mathrm{mod}}\,\Lambda}(\Pi))$

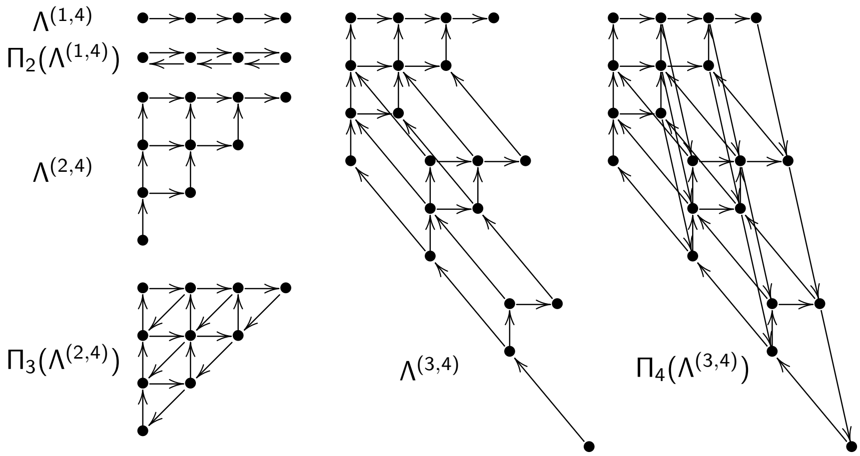
In particular,
- [Crawley-Boevey '00 for $d = 1$]

$\underline{\mathrm{mod}}\,\Pi$ is a $(d+1)$-Calabi-Yau triangulated category
- [Geiss-Leclerc-Schröer '06 for $d = 1$]

$\underline{\mathrm{mod}}\,\Pi$ has a $(d+1)$-cluster tilting object

Example of type A

$\forall\ d, n \geq 1$, $\exists\ \Lambda^{(d,n)}$: a d-representation-finite k-algebra s.t.
- $\Lambda^{(1,n)} = k\mathbb{A}_n$ and $\Lambda^{(d+1,n)} = \mathrm{End}_{\Lambda^{(d,n)}}(\Pi_{d+1}(\Lambda^{(d,n)}))$



- $\exists$ a triangle equivalence $\underline{\mathrm{mod}}\,\Pi_{d+1}(\Lambda^{(d,n)}) \simeq \mathrm{C}_{d+1}(\Lambda^{(d+1,n-1)})$

d-representation-infinite case

Setting

- Λ : a d-representation-infinite k-algebra
- Assume $\Pi := \Pi_{d+1}(\Lambda)$ is noetherian
- $e \in \Lambda$: an idempotent s.t. $\dim_k(\Pi/(e)) < \infty$ and $e\Lambda(1-e) = 0$

Theorem [Amiot-I-Reiten '15] (cf. [de Thanhoffer de Völcsey-Van den Bergh '16])

Under the above setting, let $R := e\Pi e$ and $\Lambda' := \Lambda/(e)$

- R is a Gorenstein ring
- Λ' is a finite dimensional k-algebra with $\mathrm{gl.dim}\,\Lambda' \leq d$
- $\underline{\mathrm{CM}}^{\mathbb{Z}}\,R$ has a tilting object Πe
- $\exists$ a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}\,R \simeq \mathrm{D^b}(\mathrm{mod}\,\Lambda')$
- $\underline{\mathrm{CM}}\,R$ has a d-cluster tilting object Πe
- $\exists$ a triangle equivalence $\underline{\mathrm{CM}}\,R \simeq \mathrm{C}_d(\Lambda')$

Example ($d = 1$)

Q : an extended Dynkin quiver, e : the extended vertex
$\Pi := \Pi_2(kQ)$, $R := e\Pi e$

- [Geigle-Lenzing] R is a simple singularity with $\dim R = 2$
- [Auslander] $\mathrm{CM}\,R$ has a 1-cluster tilting object Πe
- For the Dynkin quiver $Q' = Q \setminus e$, $\exists$ triangle equivalences

$\underline{\mathrm{CM}}^{\mathbb{Z}}\,R \simeq \mathrm{D^b}(\mathrm{mod}\,kQ')$ and $\underline{\mathrm{CM}}\,R \simeq \mathrm{C}_1(kQ')$

Theorem [Minamoto-Mori '11, Keller '11, Amiot-I-Reiten '15]

$\exists$ a bijection between
(1) d-representation-infinite k-algebras Λ, and
(2) $(d+1)$-Calabi-Yau k-algebras Γ of a-invariant -1, i.e.

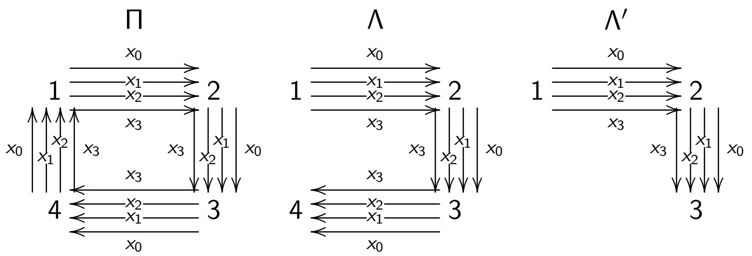
- Γ is a $\mathbb{Z}$-graded k-algebra with $\Gamma_i = 0$ for $i < 0$ and $\dim_k \Gamma_i < \infty$
- $\mathrm{proj.dim}\,(\Gamma)_{\Gamma^e} < \infty$ for $\Gamma^e := \Gamma^{\mathrm{op}} \otimes_k \Gamma$
- $\mathrm{Ext}^i_{\Gamma^e}(\Gamma, \Gamma^e) \simeq \begin{cases} 0 & i \neq d+1 \\ \Gamma(-1) & i = d+1 \end{cases}$ in $\mathrm{Mod}^{\mathbb{Z}}\Gamma^e$

The bijection is given by $\Lambda \mapsto \Pi_{d+1}(\Lambda)$, and the converse is $\Gamma \mapsto \Gamma_0$

Example (Quotient singularity) (cf. [Ueda '08])

ζ : a primitive n-th root of unity
$G := \langle \mathrm{diag}(\zeta^{a_0}, \dots, \zeta^{a_d}) \rangle \subset \mathrm{SL}_{d+1}(k)$ $(0 \le a_i < n)$
Assume $(n, a_i) = 1$ and $a_0 + \cdots + a_d = n$
$\Longrightarrow \exists$ a triangle equivalence $\underline{\mathrm{CM}}\,(k[x_0, \dots, x_d]^G) \simeq \mathrm{C}_d(\Lambda')$
for some explicit finite dimensional k-algebra Λ'

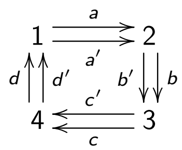
$d = 3,\ n = 4,\ a_0 = a_1 = a_2 = a_3 = 1$

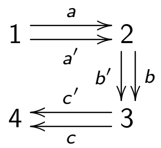


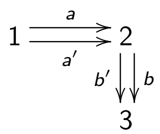
- Dimer model [Broomhead, Ishii-Ueda, Davison, Bocklandt,...]

Example ($d = 2$, Dimer model) [Nakajima '18]

R : a Gorenstein toric singularity with $\dim R = 3$
$\Longrightarrow \exists$ a triangle equivalence $\underline{\mathrm{CM}}\, R \simeq \mathrm{C}_2(\Lambda')$
for some explicit finite dimensional k-algebra Λ'

Π  $W = abcd - ab'cd' + a'b'c'd' - a'bc'd$

Λ  $abc = a'bc',\ ab'c = a'b'c'$

Λ'  $R = e_4 \Pi e_4$: the anticanonical ring of $\mathbb{P}^1 \times \mathbb{P}^1$